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RAY METHOD OF CALCULATING THE WAVE FRONT INTENSITY IN NONLINEARLY ELASTIC MATERIAL

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There is studied the behavior of the jumps of derivatives of the displacements on wave fronts of the weak discontinuity type and weak shocks being propagated on a nonlinear hyperelastic medium. There is presented the necessary information concerning the covariant differentiation with respect to time for the tensors of different structure defined on a moving surface, as well as formulas for the derivatives of certain tensor fields and for the change in geometric divergence along the rays.

Known ray methods of calculating the intensity of wave fronts [1] lead to a number of identity relationships which are satisfied by the values of the discontinuities of the derivatives satisfy on the bicharacteristics; these identities permit determination of their subsequent values by means of the jumps in the derivatives of the displacements given at the initial instant, and thereby permit the separation of the analysis of strong or weak discontinuities from the investigation of the solution at the remaining points of the space. However, the methods developed in [1] are essentially related to the linearity of the problems. Recently, methods of investigating singular surfaces, based on the use of compatibility relationships and equations significantly less sensitive to nonlinearity [2, 3] have been used extensively; in studying weak discontinuities these methods also permit obtaining the necessary identities on the bicharacteristics. In the second scheme, however, the analysis of shocks in nonlinear media is not included successfully since the fundamental role of the bicharacteristic is eliminated to a significant extent in this case. On the other hand, according to the results in [4, 9], ray representations turn out to be quite useful in the investigation of weak shocks in fluids, where the nonlinear formulation results in substantially different damping laws as compared to the acoustic laws. The method of investigating singular surfaces used in this paper is quite general and consists of using an infinite set of partial differential equations (the governing system) obtained by using different compatibility relations. In the linear case the equations of the governing system are equivalent to the transport formulas achieved by known ray methods. The governing system can be used to study different kinds of singular surfaces being propagated in different materials. For "shock" singularities of linear hyperbolic problems and for weak discontinuities of nonlinear problems, the governing system turns out to be recursive, i.e., the first M equations of this system completely describe the change in the first $M - 1$ nonzero vectors of the discontinuity, which permits finding these vectors in turn. In the case of shocks in a nonlinearly elastic medium, the recursiveness of the governing system is violated it is again restored, however, in each order of the successive approximations when finding the solution of the governing system in the form of a series in a small parameter characterizing the scale of the weak shock intensity at the initial instant. Relationships corresponding to the first approximation differ significantly from the acoustic relations [1, 10], but agree completely with those obtained earlier for weak shocks in a fluid. A different method of studying weak shocks, on the basis of using an infinite system of identities has recently been proposed in [11].

The crux of the proposed approach is demonstrated in Sect. 1 by the simplest example of a model gasdynamics equation (it should however be kept in mind that although the analysis of the model equation yields the basic features of the method and the structure of the equations which occur, two aspects of the general situation are lost here: firstly, the one-dimensionality of the problem makes the ray construction trivial, and secondly, in the case of one dependent variable the singularities in the formulation of the initial data for the governing system are not seen). The equation of motion of a hyperelastic body are presented in Sect. 2 in components referred to the initial configuration; there is a derivation in [12], for instance, from which certain notation is also borrowed. The necessary information about covariant differentiation with respect to time, of tensors of different construction defined on the moving surface is presented in Sect. 3; this formal apparatus turns out to be quite effective in simplifying the awkward equations governing the system. One of the stages in investigating weak shocks by means of the scheme proposed in this paper turns out to be equivalent to studying the propagation of weak discontinuities; the treatment of this latter problem, however, given in [2, 3] turns out to be inadequate for the purposes stated (particularly in the part concerning the introduction of rays); in this connection, a new examination of the problem of a weak discontinuity is given in Sects. 4 and 5. Moreover, the study of this latter problem specifies the form of the initial conditions for the governing system and the method introducing the ray coordinate system in the shock case. Weak shock propagation in the unperturbed domain of a hyperelastic body is examined in Sect. 6. Finally, refinements of the preceding results (concerning waves with an isolated eigenvalue of the acoustic tensor), which are needed to analyze fronts of transverse type being propagated in an undeformed domain of an isotropic nonlinearly elastic body are presented in Sect. 7.

1. The model gasdynamics problem is to solve the Cauchy problem for first order partial differential equations with two independent variables z, t and one dependent variable $v(z, t)$

$$\frac{\partial v(z, t)}{\partial t} + \frac{\partial f(v(z, t))}{\partial z} = 0 \tag{1.1}$$

The condition

$$c\,[v]_-^+ = [f(v)]_-^+ \tag{1.2}$$

where $[\;\;]_-^+$ denotes the jump in the quantity and c is the rate of displacement of the discontinuity, should be satisfied when finding piecewise-continuous solutions at points of the discontinuity.

Let the law of discontinuity motion be $z = z(t)$, where $z(t)$ is a sufficiently smooth function. Let us assume that the behavior of the solution $v(z, t)$ and its derivatives of the necessary order in the left- and right-sided semicircles of the discontinuity satisfy the conditions for applicability of the Hadamard lemma [13]; we shall henceforth call such discontinuities regular. The limit value of the discontinuous quantity will be marked by a plus (minus) sign if it approaches the discontinuity from the right (left); these limit values are clearly functions of just one independent variable, the time t, for instance. In conformity with the Hadamard lemma, we have

$$\frac{d}{dt}\frac{\partial^i v(z, t)}{\partial z \ldots \partial z_\pm} = c(t)\frac{\partial^{i+1} v(z, t)}{\partial z \ldots \partial z_\pm} + \frac{\partial^{i+1} v(z, t)}{\partial t\, \partial z \ldots \partial z_\pm} \tag{1.3}$$

Subtracting the relationship (1.3) with plus and minus signs term by term, we arrive at the compatibility relationships for the discontinuities

$$\left[\frac{\partial^{i+1} v(z, t)}{\partial t\, \partial z \ldots \partial z}\right]_-^+ = \frac{d\varkappa_i(t)}{dt} - c(t)\varkappa_{i+1}(t) \tag{1.4}$$

$$\varkappa_0(t) = [v(z, t)]_-^+, \quad \varkappa_i(t) = \left[\frac{\partial^i v(z, t)}{\partial z \ldots \partial z}\right]_-^+$$

We proceed as follows to obtain the governing system. We obtain the first equation by equating the jump on the discontinuity in the left side of (1.1) to zero and replacing $[\partial v(z, t) / \partial t]_-^+$ according to the first $(i = 0)$ of the compatibility relationships (1.4). To obtain the i-th equation of the governing system, the same procedure should be applied to the equation occurring to an i-tuple differentiation of (1.1) with respect to z, hence $[\partial^{i+1} v / \partial t \partial z \ldots \partial z]_-^+$ should be replaced according to (1.4). Expressing the limit values on the left by using $\varkappa_i$ and the limit values on the right, we obtain

$$\frac{d\varkappa_0}{dt} = \varkappa_1[c - f'(v_+ - \varkappa_0)] + [f'(v_+ - \varkappa_0) - f'(v_+)]\frac{\partial v}{\partial z_+} = \tag{1.5}$$

$$\varkappa_1[c - f'(v_+ - \varkappa_0)] + E_0\left(\varkappa_0, v_+, \frac{\partial v}{\partial z_+}\right)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{d\varkappa_i}{dt} = \varkappa_{i+1}[c - f'(v_+ - \varkappa_0)] + E_i\left(\varkappa_0, \ldots, \varkappa_i, v_+, \ldots, \frac{\partial^{i+1} v}{\partial z \ldots \partial z_+}\right)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

In the case of a shock discontinuity the system (1.5) is supplemented by the relationship resulting from (1.2)

$$c\varkappa_0 + f(v_+ - \varkappa_0) - f(v_+) = 0 \tag{1.6}$$

According to the definition of the velocity of the discontinuity, we have

$$dz(t)/dt = c \tag{1.7}$$

The relationships (1.5) —(1.7) form the governing system which the function $\varkappa_i(t)$, $z(t)$, $c(t)$ satisfy. We supplement this system by the initial conditions $\varkappa_i(t) = a_i$, $z(0) = z_*$.

Remark. 1°. Another (equivalent) system can be obtained in place of the governing system (1.5) —(1.7) by using an i-tuple differentiation with respect to t, say, in the i-th equation. Additional compatibility relationships for the discontinuities of the derivatives would hence be required.

2° Let us assume that the functions $v_+, \ldots, \partial^i v/\partial z \ldots \partial z_+, \ldots$ are known as functions of t, and let us examine a j-th order weak discontinuity. In this case the functions $\varkappa_0, \varkappa_1, \ldots, \varkappa_{j-1}$ vanish by definition, and $\varkappa_j \neq 0$ $(j \geqslant 1)$. Equation (1.6) and the first $j - 1$ equations (1.5) are hence satisfied automatically, and $c = f'(v_+)$ follows from the j-th equation. The system (1.5) is recursive in nature, i.e., any of the M first equations are closed relative to their unknowns, which affords a possibility of solving them alternately, from top to bottom. In this case the theory of ordinary differential equations substantially assures the uniqueness of the solution of the governing system. The recursive nature of the governing system is conserved for discontinuities of the shock type $(\varkappa_0 \neq 0)$ when the model equation is linear (i.e., the function f is linear). The validity of using several of the first equations of the governing system is also evident from the discussion presented, even if utilization of the remaining equations is impossible because of the inadequacy of the smoothness of the function f or the inadequate regularity of the discontinuity. In this sense, the method under consideration for the investigation of singular surfaces is not related essentially to the analyticity conditions.

3°. The functions $z(t), v_+, \ldots, \partial^i v/\partial z \ldots \partial z_+, \ldots$ are not ordinarily known in advance as functions of time. However, a slight modification permits use of the governing system in the following situation of practical importance. Let the location of the discontinuity and the value of the jumps $\varkappa_i$ as well as the fact that the solution ahead of the discontinuity is a part of the everywhere-smooth solution $v^*(z, t)$ given in advance, be known at the initial time (for instance, the function $v^*(z, t) \equiv 0$) corresponds to wave propagation in the unperturbed domain). Determine the location of the wave $z(t)$ and the functions $\varkappa_i(t)$. The substitution $v_+ = v^*(z, t)$, $\partial v(z, t)/\partial z_+ = \partial v^*(z, t)/\partial z$, etc., should be performed in the governing system in examining such a problem.

4°. Just because the equations of the governing system are nonlinear, their solutions generally exist only in a bounded time interval; however, even within the domain of existence of the solutions, the initial hypotheses of applicability of the system can be violated (for instance, if the weak discontinuity under examination is overtaken by a shock, the conditions of applicability of the Hadamard lemma are violated).

It can be seen that the governing system corresponding to a shock discontinuity and a nonlinear function f turns out to be non-recursive. The uniqueness of the solution of the Cauchy problem for the governing system is hence violated if no other additional conditions are imposed. Indeed, evolution of the shock depends substantially on the state behind the front [5]. Since only the constants a_i enter in the Cauchy problem out of all the information about the state behind the front, it is sufficient to select two different functions with identical values a_i in order to see the nonuniqueness of the solution.

However, it is convenient to use the governing system in the investigation of weak shocks. We call a shock weak if the conditions

$$z(0) = z_*, \quad \varkappa_0(0) = \varepsilon, \quad x_1(0) = a_1, \ldots, \varkappa_N(0) = a_N, \ldots \tag{1.8}$$

are satisfied at the initial time and the functions $z(t)$, $c(t)$, $\varkappa_N(t)$ can be approximated by segments of series satisfying the governing system and the initial data (1.8)

$$z(t) = \sum_{p=0}^{\infty} \varepsilon^p z_p(t), \quad c(t) = \sum_{p=0}^{\infty} \varepsilon^p c_p(t) \tag{1.9}$$

$$\varkappa_0(t) = \sum_{p=1}^{\infty} \varepsilon^p \varkappa_{0p}(t), \quad \varkappa_N(t) = \sum_{p=0}^{\infty} \varepsilon^p \varkappa_{Np}(t) \quad \text{for} \quad N \geqslant 1$$

Substituting the series (1.9) into (1.5) –(1.8), it can be noted that the governing system is recurrent in each order in ε. This assures uniqueness of the series (1.8).

Remark 5°. It can be seen that the functions $z_0(t)$, $c_0(t)$, $\varkappa_{10}(t)$ agree with the corresponding functions describing propagation of a weak first-order discontinuity.

Let us examine a weak shock being propagated in an unperturbed domain. In this case the functions $\varkappa_{01}(t)$ and $\varkappa_{10}(t)$ describing the intensity of the discontinuity v and $\partial v/\partial z$ in the lowest approximation in ε have the form

$$\varkappa_{01}(t) = \frac{1}{\sqrt{1 - f''(0)a_1 t}}, \quad \varkappa_{10}(t) = \frac{a_1}{1 - f''(0)a_1 t} \tag{1.10}$$

It follows from (1.10) that the coefficients of the series (1.9) become infinite for $f''(0)a_1 > 0$ in the finite time $1/f''(0)a_1$.

In order to obtain a certain idea concerning how the unique solution of the governing system of the form (1.9) corresponds to the state behind the shock, let us consider the function $f = v^2/2$ and the initial conditions $z(0) = 0$ and $\varkappa_0(0) = \varepsilon$ (according to the Germain–Badet stability conditions [14], only discontinuities for which $\varepsilon < 0$ should be considered), $\varkappa_1(0) = a_1$ and $\varkappa_N(0) = 0$ for $N \geqslant 2$. In this case, the series (1.9) is successfully defined completely, in particular

$$\varkappa_0(t) = \frac{\varepsilon}{\sqrt{1-a_1 t}}, \quad \varkappa_1(t) = \frac{a_1}{1-a_1 t}, \quad \varkappa_N(t) = 0 \text{ for } N \geqslant 2 \tag{1.11}$$

It can be confirmed by the method of characteristics that the state behind the wave front at the initial time $v(z, 0) = -\varepsilon - a_1 z$ corresponds to the solution (1.11), where only the initial state in the domain $-\varepsilon/a_1 < z < 0$ affects the evolution of the shock for $a_1 < 0$ (if the shock outrunning that under consideration certainly does not leave the domain $z < -\varepsilon/a_1$; the function $-\varepsilon - a_1 z$ is a Taylor series determined by the coefficients $\partial^i v / \partial z \ldots \partial z_- = -\varkappa_i(0)$. As follows from (1.11), for $a_1 < 0$ the functions $\varkappa_0(t)$ and $\varkappa_1(t)$ damp to zero (according to (1.10) the first members of the series (1.9) for the arbitrary function f and any initial data satisfying the condition $f''(0) a_1 < 0$) have the same damping nature). The asymptotic law (as $t \to \infty$) of damping of the shock intensity $\varkappa_0 \sim \text{const} \sqrt{t}$ results from (1.11). As has been shown in [15], this asymptotic damping law is quite universal, as also partly results from (1.10), (1.11) and the fact that any profile in the domain $-\varepsilon/a_1 < z < 0$ is almost linear for sufficiently small ε and fixed $a_1 < 0$.

2. At any time t the Lagrange coordinates x^1, x^2, x^3 correspond to a point x of a continuous body. In the initial state the radius-vector $\mathbf{x}(x)$ differentiable the necessary number of times, the metric tensor $x_{ij}(x)$, $x^{ij}(x)$ which is used to raise and lower the Latin indices, the bases $\mathbf{x}_i(x) = \partial \mathbf{x} / \partial x^i$, $\mathbf{x}^i = x^{ij}\mathbf{x}_j$ all correspond to the point x. The covariant derivative on the basis of the metric tensor x_{ij} is denoted by a Latin index after the vertical bar. In the deformed state, the radius vector $\mathbf{X}(x, t) = \mathbf{x} + \mathbf{u}$, where $\mathbf{u}(x, t) = u^i(x, t)\mathbf{x}_i$ is the displacement vector, corresponds to the point x.

If a hyperelastic body is characterized by a potential $\varphi(x, u_{k/l})$ and the density in the initial state $m(x)$, then the following relationships should be satisfied in the domain of twice continuously differentiable displacements (for simplicity it is assumed that there are no mass forces)

$$m \frac{\partial^2 u^i}{\partial t^2} = \left(\frac{\partial \varphi}{\partial u_{i|j}}\right)_{|j} \tag{2.1}$$

If a singular surface is propagated in the body

$$x^i = x^i(\xi^\alpha, t), \quad i = 1, 2, 3; \quad \alpha = 1, 2 \tag{2.2}$$

on which the displacements are continuous and their derivatives undergo a discontinuity, then the conditions

$$\left[\frac{\partial \varphi}{\partial u_{i|j}}\right]_-^+ n_j + \left[m \frac{\partial u^i}{\partial t}\right]_-^+ c = 0 \tag{2.3}$$

should be satisfied because of the integral law of momentum conservation.

Without deriving the relationships (2.3), let us just note that $n_j\,(\xi,\,t)$ are components of the unit normal $\mathbf{n}$ in the initial basis, and $c\,(\xi,\,t)$ is the velocity in the direction of this normal which characterizes a surface with the radius-vector

$$\xi\,(\xi,\,t) = \mathbf{x}\,(x\,(\xi,\,t)) \tag{2.4}$$

which should be distinguished from the real surface of the discontinuity which has the radius-vector $\Xi\,(\xi,\,t) = \mathbf{X}\,(x\,(\xi,\,t),\,t)$. The geometric and kinematic characteristics of both surfaces are closely related, however. Moreover, these surfaces coincide in the important particular case when the material on one side of the front is at rest.

The functions m and φ, which define the specific hyperelastic model, are assumed continuous together with all the derivatives later encountered.

3. The metric tensor $\xi_{\alpha\beta}\,(\xi,\,t)$, $\xi^{\alpha\beta}\,(\xi,\,t)$ which is used to raise and lower the Greek indices as well as to perform covariant differentiation denoted by a Greek index after a semicolon, corresponds to the surface (2.4). Let us define the covariant differentiation with respect to time of mixed tensors on a moving surface by the following relationships:

$$\frac{\delta T^{i\cdot\alpha\cdot}_{\cdot j\cdot\beta}(\xi,t)}{\delta t} = \frac{\partial T^{i\cdot\alpha\cdot}_{\cdot j\cdot\beta}}{\partial t} - v^\gamma T^{i\cdot\alpha\cdot}_{\cdot j\cdot\beta;\,\gamma} + x^i_{lm} v^l T^{m\cdot\alpha\cdot}_{\cdot j\cdot\beta} - x^m_{lj} v^l T^{i\cdot\alpha\cdot}_{\cdot m\cdot\beta} + v^\alpha_{\cdot;\,\gamma} T^{i\cdot\gamma\cdot}_{\cdot j\cdot\beta} - v^\gamma_{\cdot;\,\beta} T^{i\cdot\alpha\cdot}_{\cdot j\cdot\gamma} \tag{3.1}$$

$$v^i(\xi,t) = \frac{\partial x^i(\xi,t)}{\partial t}, \quad v^\alpha(\xi,t) = v^i(\xi,t)\, x^{\cdot\alpha}_{i\cdot}, \quad x^{i\cdot}_{\cdot\alpha}(\xi,t) = \frac{\partial x^i(\xi,t)}{\partial \xi^\alpha}$$

Here $x^i_{jk}\,(x)$ is an affinity of the second kind of the initial configuration. The operation (3.1) is covariant relative to the natural substitution $x^{i'} = x^{i'}\,(x^i)$, $\xi^{\alpha'} = \xi^{\alpha'}\,(\xi^\alpha,\,t)$ for the consideration of a moving surface [16]. In the general case the definition (3.1) neither agrees with the Thomas [17] definition of the $\delta/\delta t$ derivative nor with the Truesdell – Tupin definition [16]. However, the operation (3.1) is equivalent to the Thomas derivative for tensors having only Latin indices, and is equivalent to the Truesdell – Tupin derivative for tensors having only Greek indices, and tensors having only Latin indices which are "contractions". The tensor $T^{i\,*}_{\cdot j}\,(\xi,\,t)$ is a "contraction" of the tensor $T^i_{\cdot j}\,(x,\,t)$ if $T^{i\,*}_{\cdot j}\,(\xi,\,t) = T^i_{\cdot j}\,(x\,(\xi,\,t),\,t)$. It follows from the above that the operation (3.1) conserves the relationship obtained earlier [16, 17]:

$$\frac{\delta T^{i}_{\cdot j}{}^{*}}{\delta t} = \frac{\partial T^{i}_{\cdot j}(x, t)^{*}}{\partial t} + cn^{k}T^{i}_{\cdot j|k}(x, t)^{*} \tag{3.2}$$

$$\frac{\delta x_{ij}}{\delta t} = \frac{\delta \delta_{j}^{i}}{\delta t} = \frac{\delta x^{ij}}{\delta t} = \frac{\delta x^{i\cdot k}}{\delta t} = 0$$

$$\frac{\delta n_{i}}{\delta t} = -c_{;\alpha}x_{i\cdot}^{;\alpha}, \quad \frac{\delta \xi_{\alpha\beta}}{\delta t} = -2cb_{\alpha\beta}, \quad \frac{\delta \xi^{\alpha\beta}}{\delta t} = 2cb^{\alpha\beta}$$

Here $b_{\alpha\beta}(\xi, t) = x^{i}_{\alpha;\beta}n_{i}$ is the tensor of the second quadratic form of the surface (2.4). Using the definition (3.1), we obtain

$$\frac{\delta x^{i}_{\cdot\alpha}}{\delta t} = (cn^{i})_{;\alpha}, \quad \frac{\delta \delta^{\alpha}_{\beta}}{\delta t} = 0, \quad \frac{\delta \varepsilon_{\alpha\beta}}{\delta t} = -cb^{\omega}_{\omega}\varepsilon_{\alpha\beta}, \quad \frac{\delta \varepsilon^{\alpha\beta}}{\delta t} = cb^{\omega}_{\omega}\varepsilon^{\alpha\beta} \tag{3.3}$$

$$\frac{\delta \mathbf{x}^{*}}{\delta t} = \frac{\delta \boldsymbol{\xi}}{\delta t} = \mathbf{c} = c\mathbf{n}, \quad \frac{\delta \mathbf{x}^{*}_{i}}{\delta t} = 0, \quad \frac{\delta \boldsymbol{\xi}_{\alpha}}{\delta t} = (c\mathbf{n})_{;\alpha}$$

Here $\varepsilon_{\alpha\beta}(\xi, t)$ is the discriminant tensor of the surface (2.4).

It can be confirmed that the Leibnitz formula for the derivative of a product is valid for the operation (3.1), and commutation of differentiation and convolution is also possible.

Henceforth, for a special selection of the coordinate system ξ^{α} the object (2.4) will be considered as a two-parameter set of rays, each of which is defined by the relationship (2.4) for fixed ξ^{α}. The combination $DF^{i}(\xi, t)/Dt \equiv \delta F^{i}/\delta t + v^{\gamma}F^{i}_{\cdot;\gamma}$ agrees with the absolute derivative of the tensor $F^{i}(\xi, t)$ along the ray. The quantity $J(\xi, t) = |\xi_{\alpha\beta}(\xi, t)|^{1/2} / |\xi_{\pi\omega}|(\xi, t_{0})|^{1/2}$ is called the geometric divergence; the following relationship is valid

$$\frac{\partial J(\xi, t)}{\partial t} = J(v^{\alpha}_{;\alpha} - cb^{\alpha}_{\alpha}) \tag{3.4}$$

The definition (3.1) conserves the form of the compatibility relationships [16 – 18]. Discontinuities in the derivatives of the displacement field $u^{i}(x, t)$ can be expressed in terms of the geometric and kinematic characteristics of the singular surface, as well as in terms of vectors of the discontinuity $h^{i}_{\cdot N}(\xi, t) = [u^{i}_{\cdot|r_{1}\ldots r_{N}}]^{+}_{-} n^{r_{1}}\ldots n^{r_{N}}$ (it is clear that the numbering index N of the discontinuity vector is not tensorial) and their derivatives. In particular, we have on the surface of discontinuity of the first derivatives

$$[u^{i}_{\cdot|j}]^{+}_{-} = h^{i}_{\cdot 1}n_{j}, \quad \left[\frac{\partial u^{i}}{\partial t}\right]^{+}_{-} = -h^{i}_{\cdot 1}c \tag{3.5}$$

$$[u^{i}_{\cdot|jk}]^{+}_{-} = h^{i}_{\cdot 2}n_{j}n_{k} + 2h^{i}_{\cdot 1;\alpha}n_{(j}x_{k)\cdot}^{;\alpha} - h^{i}_{\cdot 1}b_{jk}, \quad b_{jk} \equiv b_{\alpha\beta}x_{j\cdot}^{;\alpha}x_{k\cdot}^{;\beta} \tag{3.6}$$

$$\left[\frac{\partial}{\partial t}u^{i}_{\cdot|j}\right]_{-}^{+} = -h^{i}_{\cdot 2}cn_j + \frac{\delta h^{i}_{\cdot 1}}{\delta t}n_j - (h^{i}_{\cdot 1}c)_{;\alpha}x_{j\cdot}^{;\alpha}$$

$$\left[\frac{\partial^2 u^i}{\partial t^2}\right]_{-}^{+} = h^{i}_{\cdot 2}c^2 - 2c\frac{\delta h^{i}_{\cdot 1}}{\delta t} - h^{i}_{\cdot 1}\frac{\delta c}{\delta t}$$

If derivatives starting with the second order undergo a discontinuity on the surface (2.4), then the following compatibility relationships hold:

$$[u^{i}_{\cdot|jk}]_{-}^{+} = h^{i}_{\cdot 2}n_j n_k, \quad \left[\frac{\partial}{\partial t}u^{i}_{\cdot|j}\right]_{-}^{+} = -h^{i}_{\cdot 2}cn_j, \quad \left[\frac{\partial^2 u^i}{\partial t^2}\right]_{-}^{+} = h^{i}_{\cdot 2}c^2 \tag{3.7}$$

$$[u^{i}_{\cdot|jkl}]_{-}^{+} = h^{i}_{\cdot 3}n_j n_k n_l + 3h^{i}_{\cdot 2;\alpha}n_{(j}n_k x_{l)\cdot}^{;\alpha} - 3h^{i}_{\cdot 2}n_{(j}b_{kl)} \tag{3.8}$$

$$\left[\frac{\partial u^{i}_{\cdot|jk}}{\partial t}\right]_{-}^{+} = -h^{i}_{\cdot 3}cn_j n_k + \frac{\delta h^{i}_{\cdot 2}}{\delta t}n_j n_k - 2(h^{i}_{\cdot 2}c)_{;\alpha}n_{(j}x_{k)\cdot}^{;\alpha} + h^{i}_{\cdot 2}cb_{jk}$$

$$\left[\frac{\partial^2 u^{i}_{\cdot|j}}{\partial t^2}\right]_{-}^{+} = h^{i}_{\cdot 3}c^2 n_j - 2\frac{\delta h^{i}_{\cdot 2}}{\delta t}cn_j + (h^{i}_{\cdot 2}c^2)_{;\alpha}x_{j\cdot}^{;\alpha} - h^{i}_{\cdot 2}n_j\frac{\delta c}{\delta t}$$

$$\left[\frac{\partial^3 u^i}{\partial t^3}\right]_{-}^{+} = -h^{i}_{\cdot 3}c^3 + 3c^2\frac{\delta h^{i}_{\cdot 2}}{\delta t} + 3c\frac{\delta c}{\delta t}h^{i}_{\cdot 2}$$

Using the Thomas algorithm [17], the following two equalities can be obtained which partially expand the structure of the compatibility relationships for discontinuities of the higher derivatives

$$\left[\frac{\partial^2}{\partial t^2}u^{i}_{\cdot|r_1\ldots r_N}\right]_{-}^{+}n^{r_1}\ldots n^{r_N} = h^{i}_{\cdot N+2}c^2 - 2c\cdot\frac{\delta h^{i}_{\cdot N+1}}{\delta t} - h^{i}_{\cdot N+1}\frac{\delta c}{\delta t} + L^i(h^{j}_{\cdot 0},\ldots,h^{j}_{\cdot N}) \tag{3.9}$$

$$[u^{i}_{\cdot|jkr_1\ldots r_N}]_{-}^{+}n^{r_1}\ldots n^{r_N} = h^{i}_{\cdot N+2}n_j n_k + 2h^{i}_{\cdot N+1;\alpha}x_{(j\cdot}^{;\alpha}n_{k)} - h^{i}_{\cdot N+1}b_{jk} + M^{i}_{jk}(h^{l}_{\cdot 0},\ldots,h^{l}_{\cdot N})$$

Here L^i, M^i_{jk} are differential operations of their arguments, which also depend on the goemetric and kinematic characteristics of the surface of discontinuity. The author gave an expanded analysis of the operation (3.1) earlier (*).

4. Let us turn to the construction of the governing system of equations corresponding to a singular surface in an elastic medium. To obtain the first equation, the jump of both parts on the singular surface (2.1) should be equated, and then by using the compatibility relationships mentioned, the jumps in all the quantities should be expressed in terms of the discontinuity vector and the limit values of the derivatives of

*) Grinfel'd, M. A., The $\delta/\delta t$ derivative and its properties. VINITI Dep. No. 1255 –76, 1976.

the solution from one side of the surface of discontinuity. To obtain the $(N+1)$-th equation, the same procedure must be followed with an equation obtained from (2.1) by differentiation with respect to $x^{r_1}, \ldots, x^{r_N}$ after which the result is convoluted with $n^{r_1}, \ldots, n^{r_N}$. This system should be supplemented by (2.3) treated in an analogous manner in examining a shock.

Let us show by induction that if the kinematics of the surface of weak discontinuity $(h^i_{.1}=0)$, the limit values of the derivatives of the solution $u^i(x, t)$ on one side of the wave (see Remark 3°) and the values of the vectors $h^i_{.2}, \ldots, h^i_{.M}$ are known at the initial instant, then the first M equations of the governing system will permit determination of these vectors in a certain time interval. Following the Remark 2°, we designate this property the recurrence of the governing system although the first M equations are already not closed relative to the vectors they defined (the vector $h^i_{.M+1}$) also enters). An equivalent result was first obtained in[2,3]in another interpretation.

By using the first relationship (3.7), the first equation of the governing system can be reduced to the following form

$$(mc^2x^{ik} - Q^{ik})\,h_{k2} = 0, \quad Q^{ik} = \varphi^{ijkl}n_jn_l \tag{4.1}$$
$$\varphi^{ijkl}(x, u_{m|n}) = \frac{\partial^2\varphi(x, u_{m|n})}{\partial u_{i|j}\partial u_{k|l}}$$

In the interest of reducing the writing, the arguments of the functions in the governing system equations are not written down; in this connection, it should be kept in mind that in the long run ξ^α and t are the independent variables in these equations, for instance $\varphi^{ijkl} = \lim \varphi^{ijkl}(x, u_{m/n}(x, t))$ in (4.1) as $x^i \to x^i(\xi, t)$.

Since $h_{k2} \neq 0$ on the acceleration wave (the proof is carried out for a weak discontinuity of order 2, for definiteness), this vector is the nontrivial solution of (4.1) and $c^2(\xi, t)$ corresponds to one of the eigenvalues of this system; we differentiate the other two eigenvalues by the value of the subscript L; the realness of the eigenvalues is assured by the symmetry of the acoustic tensor Q^{ik}.

Let us assume that $c^2(\xi, t) \neq c_L{}^2(\xi, t)$. The eigenvectors $e_i(\xi, t)$, $e_{iL}(\xi, t)$ correspond to the eigenvalues c^2 and $c_L{}^2$. If all the eigenvalues are distinct, we have

$$x^{ij}e_ie_{jL} = 0, \quad x^{ij}e_{i1}e_{i2} = Q \tag{4.2}$$

If $c_1{}^2 = c_2{}^2$, then the selection of the eigenvectors $e_{iL}(\xi, t)$ is subject to the second of conditions (4.2). Considering the kinematics of the singular surface, and the solution on one side of it, known, in principle the functions $c^2(\xi, t)$, $c_L{}^2(\xi, t)$, $e_i(\xi, t)$, $e_{iL}(\xi, t)$ should be considered known. Therefore, the determination of the vector $h_{i2}(\xi, t)$ reduces to determining its modulus $h(\xi, t)$ which can be done by using the second equation of the governing system. Convoluting the second equation with $e_i(\xi, t)$, after awkward manipulations using the properties of the $\delta/\delta t$ derivative (Sect. 3), it can be reduced to the form

$$2\left(\frac{\delta h}{\delta t} + \frac{1}{mc}\varphi^{ijkl}e_ie_kx_l^{;\alpha}h_{;\alpha}\right) + h\left[-cb_\alpha^\alpha + \frac{\delta \ln mc^3}{\delta t} + \right. \tag{4.3}$$

$$\left(\frac{1}{mc}\,\varphi^{ijkl}e_i e_k n_j x_l^{\cdot\alpha}\right)_{;\alpha}+\frac{1}{mc}\,\varphi^{ijkl}e_j e_k n_j x_l^{\cdot\alpha}(\ln mc^5)_{;\alpha}-$$
$$\frac{2}{mc^2}\,\varphi^{ijklmn}e_i e_k n_j n_l\,\frac{\partial u_{m|n}(x,t)}{\partial t}\Big|_{+}-2\chi ch^2=0$$
$$\chi=\frac{\varphi^{ijklmn}}{2mc^2}\,e_i e_k e_m n_j n_l n_n,\quad \varphi^{ijklmn}(x,u_{p|q})=\frac{\partial^3\varphi(x,u_{p|q})}{\partial u_{i|j}\partial u_{k|l}\partial u_{m|n}}$$

The plus sign denotes the limit value of the appropriate discontinuous quantity on that side of the singular surface where the motion is assumed known. The condition $c(\xi,t)>0$, which can always be achieved because of the selection of the normal direction, was used in obtaining (4.3).

No specific coordinate system on the singular surface has yet been set. Let us do this, namely, let us require that the relationship

$$v^{\alpha}(\xi,t)=\frac{\partial x^i(\xi,t)}{\partial t}\,x_i^{\cdot\alpha}(\xi,t)=\frac{1}{mc}\,\varphi^{ijkl}e_i e_k n_j x_l^{\cdot\alpha} \tag{4.4}$$

be satisfied in the coordinate system to be set. We shall call such a coordinate system on the surface a ray system (the question of its existence is examined in Sect. 5).

By using the properties of the $\delta/\delta t$ derivative and the relations (3.4), (4.4), the equation (4.3) in the ray coordinate system can be reduced to the following:

$$2\,\frac{\partial h(\xi,t)}{\partial t}+h\left[\frac{\partial\ln m(x(\xi,t))\,c^5(\xi,t)\,J(\xi,t)}{\partial t}-\right.$$
$$\left.\frac{2}{mc^2}\,\varphi^{ijklmn}e_i e_k n_j n_l\,\frac{\partial u_{m|n}(x,t)}{\partial t}\Big|_{+}\right]-2\chi ch^2=0 \tag{4.5}$$

This is an ordinary differential equation containing ξ^α as a parameter, which will permit determination of $h(\xi,t)$ in a certain time interval by means of the initial data $h(\xi,t_0)$.

Now, let us assume that the vectors $h^{i}_{\cdot N}$ have already been determined for $N\leqslant R$. We shall seek the vector $h^{i}_{\cdot R+1}(\xi,t)$ in the following form

$$h^{i}_{\cdot R+1}(\xi,t)=b(\xi,t)\,e^i(\xi,t)+\sum_{L=1}^{2}b_L(\xi,t)\,e^{i}_{\cdot L}(\xi,t) \tag{4.6}$$

By using the compatibility relations (3.9), the governing system equation with the number R can be reduced to the following

$$(mc^2x^{ik}-Q^{ik})h_{kR+1}+D^i(h_{k2},\ldots,h_{kR})=0 \tag{4.7}$$

Here D_i is a differential operation with respect to ξ^α, t of their arguments including also the geometric and kinematic characteristics of the surface of discontinuity and the limit values of the derivatives of the displacement on one side, which is a known function of ξ^α and t by the inductive assumption. Convoluting (4.7) with e_{iL} and using (4.2) and (4.6), we obtain

$$b_L(\xi, t) = \frac{1}{m(c^2 - c_L{}^2)} e_{iL} D^i (h_{k_s}, \ldots, h_{kR})$$

There remains to determine $b(\xi, t)$. To do this we use the governing system equation with number $R + 1$, which is reduced to

$$(mc^2 x^{ik} - Q^{ik}) h_{kR+2} = 2mc \frac{\delta h^i_{\cdot R+1}}{\delta t} + \varphi^{ijkl} (x_j^{;\alpha} n_l + x_l^{;\alpha} n_j) h_{kR+1;\alpha} + F^i (h_{k2}, \ldots, h_{kR+1}) \tag{4.8}$$

where F^i is a differential operation of the first $R - 1$ arguments, and an algebraic operation of the last. Let us convolute (4.8) with e_i; in the ray coordinate system the equation obtained will be reduced to the form

$$\partial b(\xi, t) / \partial t = G(b, \xi, t) \tag{4.9}$$

Here G is the customary function of its arguments. Therefore, (4.9) is an ordinary differential equation containing ξ^α as a parameter and permitting the determination of $b(\xi, t)$ in a certain time interval by means of the initial data. The induction is completed.

R e m a r k 6°. It is evident from the discussion presented that at the initial instant it is sufficient to know not the vectors of the discontinuity $h_{iN}(\xi, t_0)$ but just their components $h_{iN}(\xi, t_0) \cdot e^i(\xi, t_0)$.

The proof presented for the recurrence of the governing system goes over without substantial changes to the case of higher order weak discontinuities and to the case of "shock" discontinuities of linear equations.

5. The kinematics of the surface of weak discontinuity was assumed known in Sect. 4. A procedure to determine the functions $x^i = x^i(\xi^\alpha, t)$ which yield the location of the wave in a certain time interval by means of the initial data, is considered below.

We start the description of the motion of the surface of weak discontinuity with the question of the existence of a ray coordinate system thereon, i.e., a coordinate system in which condition (4.4) is satisfied. Let us assume that the surface of discontinuity is given in a certain time interval $t_0 < t < t_1$ by using continuously differentiable functions $x^i = x^{i'}(\xi^{\alpha'}, t)$, where the matrix of the metric tensor $\| \xi_{\alpha\beta}(\xi', t) \|$ is nondegenerate.

It can be shown that for a certain time interval $t_0 < t < t_2$ specific continuously differential functions $\xi^\alpha = \xi^\alpha(\xi^{\alpha'}, t)$ (having the inverses $\xi^{\alpha'} = \xi^{\alpha'}(\xi^\alpha, t)$) exist uniquely so that $\xi^1(\xi^{\alpha'}, t_0) = \xi^{1'}$, $\xi^2(\xi^{\alpha'}, t_0) = \xi^{2'}$, and the coordinate system on the surface ξ^α will be a ray system, i.e., the relationships (4.4) will be satisfied for the functions $x^i(\xi^\alpha, t) \equiv x^{i'}(\xi^{\alpha'}(\xi^\alpha, t), t)$.

In fact, the objects in the left and right sides of (4.4) are defined correctly for the functions $x^{i'}(\xi', t)$ (it is true that the relation (4.4) is not generally satisfied for them). Let $a^{\alpha'}(\xi', t)$ and $b^{\alpha'}(\xi', t)$ denote these objects, respectively. Upon introducing new mutually one-to-one coordinates on the surface $\xi^\alpha = \xi^\alpha(\xi', t)$, $\xi^{\alpha'} = \xi^{\alpha'}(\xi, t)$ the objects $a^\alpha(\xi, t)$ and $b^\alpha(\xi, t)$ defined by the same rules will be related to $a^{\alpha'}(\xi', t)$ and $b^{\alpha'}(\xi', t)$ by the relations

$$a^\alpha(\xi, t) = a^{\alpha'}(\xi'(\xi, t), t)\,\xi^\alpha_{\alpha'}(\xi'(\xi, t), t) + \frac{\partial \xi^{\alpha'}(\xi, t)}{\partial t}\,\xi^\alpha_{\alpha'}(\xi'(\xi, t), t) \tag{5.1}$$

$$b^\alpha(\xi, t) = b^{\alpha'}(\xi'(\xi, t)\,t)\,\xi^\alpha_{\alpha'}(\xi'(\xi, t), t), \quad \xi^\alpha_{\alpha'}(\xi', t) = \frac{\partial \xi^\alpha(\xi', t)}{\partial \xi^{\alpha'}}$$

For the coordinate system ξ^α to be a ray system, the objects a^α and b^α should agree; by using (5.1) and going over to the independent variables $\xi^{\alpha'}$, this condition can be written in the form of a system of equations

$$\frac{\partial \xi^\alpha(\xi', t)}{\partial t} = [a^{\alpha'}(\xi', t) - b^{\alpha'}(\xi', t)]\,\frac{\partial \xi^\alpha(\xi', t)}{\partial \xi^{\alpha'}} \tag{5.2}$$

Let us supplement this dissociating system of equations with the initial conditions

$$\xi^1(\xi^{1'}, \xi^{2'}, t_0) = \xi^{1'}, \quad \xi^2(\xi^{1'}, \xi^{2'}, t_0) = \xi^{2'} \tag{5.3}$$

As follows from the general theory [19], the problem (5.2), (5.3) has a unique solution. The determinant $|\xi_{\alpha'}{}^\alpha(\xi', t)|$ defined by this solution is continuous in t and equals one for $t = t_0$. Then for any point $\xi^{\alpha'}$ there is a neighborhood and a time interval such that the relationships $\xi^\alpha = \xi^\alpha(\xi', t)$ will be solvable for $\xi^{\alpha'}$. Inverting the discussion, we see that the functions $\xi^\alpha(\xi', t)$ actually define the ray coordinate system.

We call each space line obtained for a fixed coordinate ξ^α in the ray equation of the surface $x^i = x^i(\xi^\alpha, t)$ a ray (see Sect. 3). In some cases, the rays corresponding to a surface of weak discontinuity can be determined without reference to the solution of the problem (5.2), (5.3), but by using a certain system of ordinary differential equations.

We determine the functions $s(x, n, t)$ and $d_k(x, n, t)$ for this solution $u^i(x, t)$ of (2.1) as the corresponding value and the normalized eigenvector of the linear system of equations

$$[m(x)s^2 x^{ik}(x) - \varphi^{ijkl}(x, u_{m|n}(x, t))n_j n_l]d_k = 0 \tag{5.4}$$

These functions characterize the possible (but certainly not existing) weak discontinuities. On the surface where the second derivatives of the displacement undergo a discontinuity, the first derivatives of the functions s and d_k will undergo discontinuity. Let us make the additional assumption that the branch of $s(x, n, t)$ corresponding to the weak discontinuity under investigation is isolated from the other branches to the whole range of the independent variables (henceforth, only this branch will be understood as s) in this case the following relationships, related to the "lemma on bicharacteristic directions" [19] are valid

$$\frac{\partial s}{\partial x^q} = \frac{1}{2ms}\frac{\partial \varphi^{ijkl}(x, u_{m|n}(x, t))}{\partial x^q} n_j n_l d_i d_k - \frac{s^2}{2m}\frac{\partial m}{\partial x^q} \tag{5.5}$$
$$\frac{\partial s}{\partial n_q} = \frac{1}{ms}\varphi^{ijkq} n_j d_i d_k$$

To obtain (5.5), we should substitute $s = s(x, n, t)$ and $d_k = d_k(x, n, t)$ into (5.4), differentiate the identity obtained with respect to the appropriate argument and convolute the result with $d_i(x, n, t)\cdot$ (it should be kept in mind that the identity of the relation being differentiated is generally violated upon replacing $x^{ij} n_i n_j$ by one, which it will be convenient to do in analyzing fluids and other isotropic media. However, this can involve violation of the relation (5.5); in this case, small changes in the reasoning are needed, which we do not examine).

Let us differentiate the equation of the surface of weak discontinuity the field of its unit normals with respect to time in the ray coordinate system (for the sake of brevity, the subsequent computations in the paper are performed in the Lagrange coordinate system, which is affine in the initial configuration)

$$\frac{\partial x^p(\xi, t)}{\partial t} = \frac{\partial x^q(\xi, t)}{\partial t}(x^{p\cdot}_{\cdot\alpha} x^{\cdot\alpha}_{q\cdot} + n^p n_q) = v^\alpha x^p_{\cdot\alpha} + cn^p = \tag{5.6}$$
$$\frac{1}{mc}\varphi^{ijkl} e_i e_k n_j x_l^{\cdot\alpha} x^{p\cdot}_{\cdot\alpha} + cn^p = \left.\frac{\partial s(x, n, t)}{\partial n_p}\right|_+$$
$$\frac{\partial n_p(\xi, t)}{\partial t} = \frac{\delta n_p}{\delta t} + v^\alpha n_{p;\alpha} = -c_{;\alpha} x^{\cdot\alpha}_{p\cdot} + \left.\frac{\partial s}{\partial n_q}\right|_+ x_q^{\cdot\alpha} n_{p;\alpha} =$$
$$-\left.\frac{\partial s(x, n, t)}{\partial x^q}\right|_+ (\delta_p{}^q - n^q n_p)$$

Since the derivatives of the function $s(x, n, t)$ undergo a discontinuity on the wave front, the appropriate limit value is explicitly indicated by a plus. The relations

$$c(\xi, t) = s(x(\xi, t), n(\xi, t), t), e_i(\xi, t) = d_i(x(\xi, t), n(\xi, t), t) \tag{5.7}$$
$$x^{p\cdot}_{\cdot\alpha} x^{\cdot\alpha}_{q\cdot} = \delta^p_q - n^p n_q, \quad \frac{\partial s(x, n, t)}{\partial n_l} n_l = s(x, n, t)$$

were used to obtain (5.6).

According to Sect. 3 and the definition of a ray, (5.6) and (5.7) characterize the change in the functions x^p, n_p along the rays. Construction of the domain of continuous differentiability of the function $s(x, n, t)$ does not generally permit using these equations to construct the rays and the surface of discontinuity. However, such a possibility is manifest if the solution before the discontinuity is given as a part of some twice continuously-differentiable solution defined in the whole space (see Remark 3°).

6. By analogy with (4.4), we define the ray coordinate system on the surface in the shock case by the condition

$$mcv^{\alpha}(\xi, t) = \varphi^{ijkl}_{+} n_j x_{l\cdot}^{\cdot\alpha} e_{i1} e_{k1}, \quad e_{i1} \equiv \frac{h_{i1}}{|h_{i1}|} \tag{6.1}$$

If the material in front of the shock is in the undeformed state, then by using (3.5)—(3.8), the relationship (2.3), the first two equations of the governing system, and the relationship (6.1), can be reduced to the following:

$$[\varphi^{ij}(x, -h_{k1}n_l) - \varphi^{ij}(x, 0)]\, n_j + mc^2 h^{i}_{\cdot 1} = 0, \quad \varphi^{ij} = \frac{\partial \varphi(x, u_{k|l})}{\partial u_{i|j}} \tag{6.2}$$

$$[mc^2 x^{ik} - \varphi^{ijkl}(x, -h_{p1}n_q)\, n_j n_l]\, h_{k2} - 2mc\frac{\delta h^{i}_{\cdot 1}}{\delta t} - m\frac{\delta c}{\delta t} h^{i}_{\cdot 1} +$$
$$\varphi^{ijkl}(x, -h_{p1}n_q)(h_{k1}b_{jl} - 2h_{k1;\alpha} n_{(j} x_{l)\cdot}^{\cdot\alpha}) + \frac{\partial \varphi^{ij}(x, -h_{k1}n_l)}{\partial x^j} = 0$$

$$[mc^2 x^{ik} - \varphi^{ijkl}(x, -h_{p1}n_q)\, n_j n_l]\, h_{k3} - 2mc\frac{\delta h^{i}_{\cdot 2}}{\delta t} -$$
$$m\frac{\delta c}{\delta t} h^{i}_{\cdot 2} + \varphi^{ijkl}(x, -h_{p1}n_q)(h_{k2}b_{jl} - 2h_{k2;\alpha} n_{(j} x_{l)\cdot}^{\cdot\alpha}) +$$
$$\frac{\partial^2 \varphi^{ij}(x, -h_{p1}n_q)}{\partial x^j \partial x^r} n^r - \frac{\partial \varphi^{ijkl}(x, -h_{p1}n_q)}{\partial x^j}(h_{k2}n_l + h_{k1;\alpha} x_{l\cdot}^{\cdot\alpha}) -$$
$$\frac{\partial \varphi^{ijkl}(x, -h_{p1}n_q)}{\partial x^r}(h_{k2}n_j n_l + 2h_{k1;\alpha} n_{(j} x_{l)\cdot}^{\cdot\alpha} - h_{k1}b_{jl})\, n^r +$$
$$m_{|r} n^r \left(h^{i}_{\cdot 2} c^2 - 2c\frac{\delta h^{i}_{\cdot 1}}{\delta t} - h^{i}_{\cdot 1}\frac{\delta c}{\delta t}\right) - \varphi^{ijklmn}(x, -h_{p1}n_q)(h_{k2}n_j n_l +$$
$$2h_{k1;\alpha} n_{(j} x_{l)\cdot}^{\cdot\alpha} - h_{k1} b_{jl})(h_{m2}n_n + h_{m1;\alpha} x_{n\cdot}^{\cdot\alpha}) = 0$$
$$mc\frac{\partial x^i(\xi, t)}{\partial t} x_{i\cdot}^{\cdot\alpha} = \varphi^{ijkl}(x, -h_{p1}n_q)\, n_{(j} x_{l)\cdot}^{\cdot\alpha} e_{i1} e_{k1}$$

We supplement the governing system by the initial conditions (see Remark 6°)

$$h^{i}_{\cdot 1}(\xi, t_0)\, e_{i1}(\xi, t_0) = \varepsilon A_1(\xi) \tag{6.3}$$
$$h^{i}_{\cdot N}(\xi, t_0)\, e_{i1}(\xi, t_0) = A_N(\xi), \; N \geqslant 2; \; x^i(\xi, t_0) = x_*^{\,i}(\xi)$$

We call a shock weak if the function describing it can be approximated by segments of the following series (satisfying the governing system with the initial conditions (6.3)):

$$x^i(\xi,t)=\sum_{p=0}^{\infty}x^i_{\cdot p}(\xi,t)\,\varepsilon^p,\quad h^i_{\cdot 1}(\xi,t)=\sum_{p=1}^{\infty}h^i_{\cdot 1p}(\xi,t)\,\varepsilon^p \tag{6.4}$$
$$h^i_{\cdot N}(\xi,t)=\sum_{p=0}^{\infty}h^i_{\cdot Np}(\xi,t)\,\varepsilon^p,\quad N\geqslant 2$$

The form of the series (6.4) is selected in a computation such that the discontinuity in the first derivatives would tend to zero as $\varepsilon\to 0$ while the discontinuities in the higher order derivatives remain finite. Substituting the series (6.4) into the governing system equations, it can be seen that this system turns out to be recurrent in the sense of Sect. 4 in each order in ε. Comparing terms in identical powers in ε in (6.2), we find: 1) the vectors h_{i11}, h_{i20} are null vectors of the matrix $A^{ik}=m(x_0)\,c_0^2x^{ik}-\varphi^{ijkl}(x_0,0)\,n_{j0}n_{l0}$ (objects referring to the surface $x^i=x^i_{\cdot 0}(\xi,t)$) are marked with the zero), 2) the equations which the functions $x^i_{\cdot 0}$, c_0, n_{i0}, h_{i20} satisfy agree with the equations of Sect. 4 describing acceleration wave propagation in the rest domain (see Remark 5°).

Let us assume that the isolated eigenvalue c_0^2 with the unit normalized null-vector of the matrix $A^{ik}-r_i(\xi,t)$ corresponds to the wave under consideration. Then we have $h_{i20}(\xi,t)=h(\xi,t)\,r_i(\xi,t)$, $h_{i11}(\xi,t)=\eta(\xi,t)\,r_i(\xi,t)$. The following equations result for the intensities $h(\xi,t)$ and $\eta(\xi,t)$ from the governing system:

$$2\frac{\partial h(\xi,t)}{\partial t}+h\frac{\partial\ln m(x_0)\,c_0^5J_0(\xi,t)}{\partial t}-2c_0\nu(\xi,t)\,h^2=0 \tag{6.5}$$
$$2\frac{\partial\eta(\xi,t)}{\partial t}+\eta\frac{\partial\ln m(x_0)\,c_0^3J_0(\xi,t)}{\partial t}-c_0\nu(\xi,t)\,h\eta=0$$
$$\nu(\xi,t)\equiv\frac{1}{2m(x_0)\,c_0^2}\,\varphi^{ijklmn}(x_0,0)\,n_{j0}n_{l0}n_{n0}r_ir_kr_m$$
$$\eta(\xi,t_0)=A_1(\xi),\quad h(\xi,t_0)=A_2(\xi)$$

which permit finding the functions η and h, and describing the change in the jumps in the first and second derivatives of the displacement in a lower approximation in ε.

Remark 7°. Equations (6.5) admit an integral since

$$\frac{d}{dt}\frac{h^2}{\eta^4m(x_0)\,c_0J_0}=0$$

It can be shown that this integral exists for an arbitrary state of the medium ahead of the front and, in combination with (4.5), permits a study of the behavior of weak shocks even in this case.

Let us consider a plane, weak shock being propagated in the rest domain of a homogeneous medium. In this case, we find from (6.5)

$$\eta=\frac{A_1}{\sqrt{1+c_0\nu A_2(t-t_0)}},\quad h=\frac{A_2}{1+c_0\nu A_2(t-t_0)} \tag{6.6}$$

which can be converted into a more graphic form in complete analogy with (1.10). To do this, we let $\sigma(\xi, t)$ denote the jump in density of the surface force during passage through the wave front. To the accuracy of second order terms in ε, we have $\sigma(\xi, t) \approx \sigma(\xi, t)\, \mathbf{r}(\xi, t)$ on the weak shock surface, where $\sigma(\xi, t) = \varepsilon\eta(\xi, t)\, m(x_0)\, c_0^2$, $\mathbf{r}(\xi, t) = r^i(\xi, t)\, \mathbf{x}_i$. Having defined the shock length by the formula $L(\xi, t) = |h_{i1}(\xi, t)| / |h_{i2}(\xi, t)|$, we obtain $L(\xi, t) \approx \Lambda(\xi, t) = \varepsilon\eta / h$ to second order accuracy in ε in the case of a weak shock.

The relationships (6.6) can be written in terms of σ and Λ

$$\Lambda(t) = \Lambda_0 \left[1 + \frac{\sigma_0 \nu (t - t_0)}{\Lambda_0 c_0 m}\right]^{1/2}, \quad \sigma(t) = \sigma_0 \left[1 + \frac{\sigma_0 \nu (t - t_0)}{\Lambda_0 c_0 m}\right]^{-1/2} \tag{6.7}$$

$$\Lambda_0 = \Lambda(t_0), \quad \sigma_0 = \sigma(t_0)$$

which agree, substantially, with the formulas describing weak shocks in a fluid [7]. For sufficiently large t and $\nu \neq 0$, the following asymptotic intensity damping laws can be obtained from (6.7) and analogous relationships for cylindrical and spherical waves (for plane, cylindrical, and spherical waves, respectively):

$$\sigma \sim \text{const}\ t^{-1/2}, \quad \sigma \sim \text{const}\ t^{-3/4}, \quad \sigma \sim \text{const}\ (t \sqrt{\ln t})^{-1} \tag{6.8}$$

which is also in complete agreement with the damping laws found earlier for weak shocks in a fluid [4, 5, 7, 8].

7. An isotropic hyperelastic material is defined by the fact that the potential φ is a function of the three principal invariants of the strain tensor, as well as of the Lagrange coordinates in the inhomogeneous body case. Weak discontinuities propagated in the undeformed domain of an isotropic elastic body will be either longitudinal $(h_{i2} = hn_2)$ or transverse $(h_{i2}n^i = 0)$ [20, 21]. An isolated eigenvalue of the acoustic tensor corresponds to the longitudinal wave, and therefore, the conditions of Sect. 4.5 are satisfied. In the case of a weak shock of longitudinal type, the quantity $\nu(\xi, t)$ is generally not zero, so that such waves damp out in conformity with (6.8).

A double eigenvalue of the acoustic tensor corresponds to a discontinuity of transverse type, hence (4.1) does not permit determination of the position of the vector h_{k2} in the tangent plane. The position of this vector, exactly as its absolute value, is determined successfully from the next equation of the governing system which admits of two independent corollaries in this case. It follows from (4.4) that in the case under consideration the rays turn out to be orthogonal to the successive positions of the front, and the above-mentioned corollaries can be obtained by convoluting the second equation of the governing system with the field of normals and binormals to the rays. Calculations very similar to those in [22] show that the intensity of the discontinuity in the second derivatives of the displacement $h(\xi, t)$ and the angle $\theta(\xi, t)$ between the vector of the discontinuity $h_{i2}(\xi, t)$ and the principal normal of the ray will satisfy the relations

$$\frac{\partial h^2 J m c_s^3(\xi, t)}{\partial t} = 0, \quad \frac{\partial \theta(\xi, t)}{\partial t} = \frac{c_s}{T} \tag{7.1}$$

Here $c_s(x)$ is the propagation velocity of weak discontinuities of transverse type in the undeformed domain of an isotropic nonlinearly elastic body, and $T(\xi, t)$ is the radius of curvature of the ray ξ^α. The relations (7.1) agree with the formulas of classical linear elasticity theory [1, 22] (this was noted in [20] with respect to the intensity of the discontinuity).

In the case of weak shocks of transverse type being propagated in the undeformed domain of an isotropic nonlinearly elastic body, collinearity of the vectors h_{k11}, h_{k20} already does not result from the governing system, but just $h_{k11}(\xi, t) = \eta(\xi, t) r_k(\xi, t)$ and $h_{k20}(\xi, t) = h(\xi, t) q_k(\xi, t)$, where r_k and q_k are vectors orthogonal to the unit normal $n_{k0}(\xi, t)$ to the surface $x^i = x_0^i(\xi, t)$. The governing system results in this case in the equation

$$2m(x_0) c_s \frac{\partial \eta(\xi, t)}{\partial t} + \eta m(x_0) c_s \frac{\partial \ln J_0 m(x_0) c_s^3(x_0)}{\partial t} - \varphi^{ijklmn}(x_0, 0) n_{j_0} n_{l_0} n_{n_0} \left(r_i q_k r_m - \frac{1}{2} q_i r_k r_m\right) \eta h = 0 \tag{7.2}$$

Since $\varphi^{ijklmn}(x, 0)$ is a linear combination of terms of the form $k(x) x^{ij} x^{kl} x^{mn}$ with different combinations of the superscripts in the case of an isotropic medium, the last term in (7.2) vanishes. The law of intensity variation of a weak shock of transverse type consequently turns out to be exactly the same as in acoustic theory [1, 10]. As is known, according to the acoustic theory, the damping rate is slower than that described by (6.8).

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REFERENCES

1. Alekseev, A. S., Babich, V. M., and Gel'chinskii, B. Ia., Ray method of calculating the wave front intensity, IN: Questions of the Dynamic Theory of Seismic Wave Propagation, No. 5, Izd, Leningrad. Univ., 1961.

2. Wright, T. W., Acceleration waves in simple elastic materials, Arch. Ration. Mech. and Analysis, Vol. 50, No. 4, 1973.

3. Brown, M., Beschleunigungswellen in anisotropen hyperelastischen Stoffen, Acta Mech., Vol. 19, No. 3-4, 1974.

4. L a n d a u, L. D., On shock waves at large distances from the place of their generation, P.M.M. Vol. 9, No. 4, 1945.

5. C o u r a n t, R, and F r i e d r i c h s, K., Supersonic Flow and Shockwaves, Izd. Inostr. Lit. Moscow, 1950.

6. G u b k i n, K. E. Propagation of discontinuities in sound waves, PMM Vol, 22, No. 4, 1958.

7. R y z h o v, O. S., Damping of shock waves in inhomogeneous media, Prikl. Mekhan. Tekhn. Fiz., No. 2, 1961.

8. W h i t h a m, G. B., Linear and Non-linear waves, J. Wiley and Sons, 1974.

9. P r a s a d, P., Approximation of the perturbation equations of a quasilinear hyperbolic system in the neighborhood of a bicharacteristic, J. Math. Analysis and Appl, Vol. 50, No. 3, 1975.

10. K e l l e r, J. B., Geometrical acoustics. 1. The theory of weak shock waves, J. Appl. Phys., Vol. 25, No. 8, 1954.

11. M a s l o v, V. P., Weak shock waves propagation in an isentropic inviscid gas, Modern Problems of Mathematics, Vol. 8, 1977.

12. M o v c h a n, A. A., On the stability of continuous body motion. Lagrange theorem and its inversion, Inzh. Sb, Vol. 29, 1960.

13. S m i r n o v, V. I., Course in Higher Mathematics, Vol. 4, (English translation), Pergamon Press, Book No. 10210, 1964.

14. R o z h d e s t v e n s k i i, B. L., and I a n e n k o, N. N., Systems of Quasilinear Equations, "Nauka", Moscow, 1968.

15. L a x, P. D., Hyperbolic systems of conservation laws. 2. Communs. Pure and Appl. Math., Vol. 10, No. 4, 1957.

16. T r u e s d e l l, C. A. and T o u p i n, R. A., The classical field theories, Handbuch Physik, Vol. 3/1, Springer, 1960.

17. T h o m a s, T., Plastic Flow and Rupture in Solids, "Mir", Moscow, 1964.

18. S u h u b i, E. S., The growth of acceleration waves of arbitrary form in deformed hyperelastic materials, Internat. J. Engng Sci., Vol. 8, No. 8, 1970.

19. C o u r a n t, R., Partial Differential Equations, New York, Wiley–Interscience, 1962.

20. J u n e j a, B. L. and N a r i b o l i, G. A., Growth of acceleration waves in an unstrained nonlinear isotropic elastic medium. Internat. J. Nonlinear Mech., Vol. 5, 1970.

21. C h e n, P. J., Growth and decay of waves in solids. Handbuch Phys. a/3, Vol. 6, Springer Verlag, 1973.

22. Levin, M. A., and Rytov, S. M., On the passage to the geometric approximation in elasticity theory, Akust. Zh., Vol. 2, No. 2, 1956.

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